MAT185 – Linear Algebra
Assignment 1

Instructions:

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3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

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I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
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- I have not used generative AI in writing this assignment.
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Question 1:

In this problem, you will prove that the elementary operations you learnt in ESC103 do not change the set of solutions of a system of linear equations. Consider the four linear systems

$$\mathcal{A}: \begin{cases} 3x + 2y + 2z = 9 \\ 11x + 7y + 3z = 15 \\ 3x + 2y + z = 5 \end{cases} \qquad \mathcal{B}: \begin{cases} 3x + 2y + 2z = 9 \\ 11x + 7y + 3z = 15 \\ (3+11\beta)x + (2+7\beta)y + (1+3\beta)z = 5 + 15\beta \end{cases}$$

$$\mathcal{C}: \begin{cases} 3x + 2y + 2z = 9 \\ 11\alpha x + 7\alpha y + 3\alpha z = 15\alpha \\ 3x + 2y + z = 5 \end{cases} \qquad \mathcal{D}: \begin{cases} 11x + 7y + 3z = 15 \\ 3x + 2y + 2z = 9 \\ 3x + 2y + z = 5 \end{cases}$$

Above, x, y, and z are all **variables**. The parameters α and β are real numbers.

Let $A \subseteq \mathbb{R}^3$ be the **set of solutions** of the linear system $\mathcal{A}$. A point $(a, b, c) \in A$ means that

$$\mathcal{A}: \begin{cases} 3a + 2b + 2c = 9 & \textbf{TRUE} \\ 11a + 7b + 3c = 15 & \textbf{TRUE} \\ 3a + 2b + c = 5 & \textbf{TRUE} \end{cases}$$

Because a, b, and c are all **real numbers**, equations involving them are either true (as in $1 + 1 = 2$ is true) or false (as in $1 + 1 = 3$ is false). And so, (a, b, c) is in A (is a solution of $\mathcal{A}$) if all three equations are true when the variables x, y, and z take on the values a, b, and c respectively. If there are no points (a, b, c) for which the three equations are all true then $\mathcal{A}$ has no solutions and $A = \varnothing$. *Note: this does not contradict the statement that $A \subseteq \mathbb{R}^3$ because $\varnothing \subseteq \mathbb{R}^3$.*

Pro tip: If you have an equation where the left-hand side and right-hand side are real numbers (or are numbers in any field) then the equation $LHS = RHS$ is **TRUE** if and only if $LHS - RHS = 0$ and is **FALSE** if and only if $LHS - RHS \neq 0$.

Finally, there is the question of how you could show that two sets, A and B, are equal.

- If you want to prove that $A = B$ and $A = \varnothing$ then you need to prove that $B = \varnothing$.
- If you want to prove that $A = B$ and $B = \varnothing$ then you need to prove that $A = \varnothing$.
- If you want to prove that $A = B$ and neither A nor B are nonempty then you can do this by proving
  - if $a \in A$ then $a \in B$ (this proves $A \subseteq B$) **and**
  - if $b \in B$ then $b \in A$ (this proves $B \subseteq A$).

If you have sets A and B and you don't know whether or not they are empty or nonempty, you have do all three steps above.

For the curious: The third case is related to another common proof technique: if you want to prove that two real numbers are equal, $a = b$, you first use one idea/approach to prove $a \leq b$ and a different idea/approach to prove $a \geq b$. This technique is jarring the first few times one sees it. This approach works for any totally-ordered field; it doesn't work for the complex numbers, for example.

(a) Let A be the set of solutions of $\mathcal{A}$ and B be the set of solutions of $\mathcal{B}$. Prove that $A = B$. *Note: you don't know whether or not $\mathcal{A}$ or $\mathcal{B}$ have any solutions. This means that you must address all three possible cases: $A = \varnothing$, $B = \varnothing$, and neither A nor B are the empty set.*

Please start your answer on the next page, not on this one. The grader will not look at this page. You can continue your answer on the top of page 4, if needed.

case 1: $A = \varnothing$

If $A = \varnothing$, then at least one of the equations of $\mathcal{A}$ are inconsitent.

Expanding and rewriting the third equation in $\mathcal{B}$

$(3+11\beta)x + (2+7\beta)y + (1+3\beta)z = 5 + 15\beta$

$3x + 11\beta x + 2y + 7\beta y + z + 3\beta z = 5 + 15\beta$ (MIII)

$\beta(11x + 7y + 3z) + (3x + 2y + z) = 5 + 15\beta$ (MIII) ... ①

Since the second equation of $\mathcal{B}$ is: $11x + 7y + 3z = 15$

① can be written as: $15\beta + (3x+2y+z) = 5 + 15\beta$

Since β is a real number: $3x + 2y + z = 5$

Rewriting $\mathcal{B}$:

$$\mathcal{B}: \begin{cases} 3x + 2y + 2z = 9 \ldots \text{B1} \\ 11x + 7y + 3z = 15 \ldots \text{B2} \\ 3x + 2y + z = 5 \ldots \text{B3} \end{cases}$$

Notice that these are the same as the equations in $\mathcal{A}$.

$$\mathcal{A}: \begin{cases} 3x + 2y + 2z = 9 \ldots \text{A1} \\ 11x + 7y + 3z = 15 \ldots \text{A2} \\ 3x + 2y + z = 5 \ldots \text{A3} \end{cases}$$

Thus, if A1 is false, then B1 is also false as they are the same equation, and so both $\mathcal{A}$ is inconsistent and $\mathcal{B}$ is inconsistent, therefore $A = \varnothing$, $B = \varnothing$.

If A2 is false, then B2 is false as they are the same equation, and thus $\mathcal{A}$ is inconsistent and $\mathcal{B}$ is inconsistent. Therefore $A = \varnothing$, $B = \varnothing$.

If A3 is false, then B3 is also false assuming B2 is true. Thus, $\mathcal{A}$ is inconsistent and $\mathcal{B}$ is inconsistent, leading to $A = \varnothing$ and thus $B = \varnothing$. ∎

case 2: $B = \varnothing$

As seen in case 1, $\mathcal{B}$ can be rewritten to:

$$\mathcal{B}: \begin{cases} 3x + 2y + 2z = 9 \ldots \text{B1} \\ 11x + 7y + 3z = 15 \ldots \text{B2} \\ 3x + 2y + z = 5 \ldots \text{B3} \end{cases}$$

Which are the same equations as $\mathcal{A}$.

$$\mathcal{A}: \begin{cases} 3x + 2y + 2z = 9 \ldots \text{A1} \\ 11x + 7y + 3z = 15 \ldots \text{A2} \\ 3x + 2y + z = 5 \ldots \text{A3} \end{cases}$$

Assuming that $B = \varnothing$, then at least one of the equations in B are inconsistent with the system.

If B1 is inconsistent, then A1 also has to be inconsistent since A1 and B1 are the same equation. Thus $A = \varnothing$.

If B2 is inconsistent, then A2 also has to be inconsistent since A2 and B2 are the same equation. Thus $A = \varnothing$.

If B2 is consistent and B3 is inconsistent, then A3 is also inconsistent since A3 and B3 have the same equation. Thus $A = \varnothing$.

However, if B2 is inconsistent, A2 is inconsistent, still leading to $A = \varnothing$.

Therefore, if $B = \varnothing$, $A = \varnothing$. ∎

case 3: both A and B are nonempty

Rewriting the equations in A, the system of equations can be solved:

$$\left[\begin{array}{ccc|c} 3 & 2 & 2 & 9 \\ 11 & 7 & 3 & 15 \\ 3 & 2 & 1 & 5 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2/3 & 2/3 & 3 \\ 11 & 7 & 3 & 15 \\ 3 & 2 & 1 & 5 \end{array}\right] \xrightarrow{R_2 - 11R_1} \left[\begin{array}{ccc|c} 1 & 2/3 & 2/3 & 3 \\ 0 & -1/3 & -13/3 & -18 \\ 3 & 2 & 1 & 5 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2/3 & 2/3 & 3 \\ 0 & -1/3 & -13/3 & -18 \\ 0 & 0 & -1 & -4 \end{array}\right] \xrightarrow{-3R_2} \left[\begin{array}{ccc|c} 1 & 2/3 & 2/3 & 3 \\ 0 & 1 & 13 & 54 \\ 0 & 0 & 1 & -4 \end{array}\right] \xrightarrow{R_1 - \frac{2}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -8 & -33 \\ 0 & 1 & 13 & 54 \\ 0 & 0 & -1 & -4 \end{array}\right]$$

Continuing:

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & -33 \\ 0 & 1 & 13 & 54 \\ 0 & 0 & -1 & -4 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|c} 1 & 0 & -8 & -33 \\ 0 & 1 & 13 & 54 \\ 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{R_1 + 8R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 13 & 54 \\ 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{R_2 - 13R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$\therefore$ solutions are $x = -1, y = 2, z = 4$

Since as proven in case 1, $\mathcal{A}$ and $\mathcal{B}$ have the same solutions, therefore since $x = -1, y = 2, z = 4$ is a set of solutions in A, then they also have to be a set of solutions in $\mathcal{B}$.

Proof $x = -1, y = 2, z = 4$ is a solution to $\mathcal{B}$:

B1. $3(-1) + 2(2) + 2(4) = 9$ $\therefore$ LHS = RHS

B2. $11(-1) + 7(2) + 3(4) = 15$ $\therefore$ LHS = RHS

B3. $(3+11\beta)(-1) + (2+\beta)(2) + (1+3\beta)(4) = 5 + 15\beta$ $\therefore$ LHS = RHS

Thus, $A \subseteq B$

Similarly, as $x = -1, y = 2, z = 4$ is a set of solutions for $\mathcal{B}$, and are also a set of solutions for $\mathcal{A}$, $B \subseteq A$.

Proof $x = -1, y = 2, z = 4$ is a solution to $\mathcal{A}$:

A1. $3(-1) + 2(2) + 2(4) = 9$ $\therefore$ LHS = RHS

A2. $11(-1) + 7(2) + 3(4) = 15$ $\therefore$ LHS = RHS

A3. $3(-1) + 2(2) + 4 = 5$ $\therefore$ LHS = RHS

Thus, $B \subseteq A$

Since the set of solutions to $\mathcal{A}$ is also the solution to $\mathcal{B}$, $A \subseteq B$. Since the set of solutions to $\mathcal{B}$ is also the solution to $\mathcal{A}$, $A \subseteq B$. Thus $A = B$ and neither A nor B are nonempty. ∎

You can continue your answer here. If you do so, please make it clear which of the three cases you're continuing!

(b) Let A be the set of solutions of $\mathcal{A}$ and C be the set of solutions of $\mathcal{C}$.

i. The system $\mathcal{C}$ has a parameter $\alpha \in \mathbb{R}$. Under what conditions on α is $A \subseteq C$? *No justification needed, just one complete sentence.*

Under the condition that $\alpha \neq 0$, $\alpha \in \mathbb{R}$, $A \subseteq C$.

ii. Under what conditions on α is $C \subseteq A$? *No justification needed, just one complete sentence.*

Under the condition that $\alpha \in \mathbb{R}$, $C \subseteq A$.

iii. Assuming the condition of part ii. holds, prove that $C \subseteq A$. *You may assume that $C \neq \varnothing$; we'll assume that if you can address the $C = \varnothing$ case based on your methods in part (a).*

Since $C \neq \varnothing$ (assumption given in question) and $A \neq \varnothing$ from part a and since for any $\alpha \in \mathbb{R}$, the solutions of C are contained in A
The reason the solutions of C are contained in A is because the equation in $\mathcal{C}$ is simply a scalar multiple of the equations in $\mathcal{A}$ by MI (multiply the third equation of $\mathcal{C}$ by $\frac{1}{\alpha}$ to get the same equations as in $\mathcal{A}$).

Therefore, $C \subseteq A$. ∎

Note that since not all α makes $A \subseteq C$, only $C \subseteq A$ is true.

(c) Let D be the set of solutions of $\mathcal{D}$. Prove that $A = D$. *You may assume that both A and D are nonempty; we'll assume you can address the case where either of them is the empty set based on your work in part (a).*

This part is worth zero points. Your proof will not be graded. If you got stuck, or wrote a proof you aren't quite sure of, please post to piazza or come to office hours.

YOU MUST UPLOAD THIS PAGE EVEN IF YOU WROTE NOTHING ON IT.

Rewriting D:

$$\left[\begin{array}{ccc|c} 11 & 7 & 3 & 15 \\ 3 & 2 & 2 & 9 \\ 3 & 2 & 1 & 5 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 11 & 7 & 3 & 15 \\ 1 & 2/3 & 2/3 & 3 \\ 3 & 2 & 1 & 5 \end{array}\right] \xrightarrow{R_1 - 11R_2} \left[\begin{array}{ccc|c} 0 & -1/3 & -13/3 & -18 \\ 1 & 2/3 & 2/3 & 3 \\ 3 & 2 & 1 & 5 \end{array}\right] \xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|c} 0 & -1/3 & -13/3 & -18 \\ 1 & 2/3 & 2/3 & 3 \\ 0 & 0 & -1 & -4 \end{array}\right]$$

Continued: $$\left[\begin{array}{ccc|c} 0 & -1/3 & -13/3 & -18 \\ 1 & 2/3 & 2/3 & 3 \\ 0 & 0 & -1 & -4 \end{array}\right] \xrightarrow{-3R_1} \left[\begin{array}{ccc|c} 0 & 1 & 13 & 54 \\ 1 & 2/3 & 2/3 & 3 \\ 0 & 0 & -1 & -4 \end{array}\right] \xrightarrow{R_2 - \frac{2}{3}R_1} \left[\begin{array}{ccc|c} 0 & 1 & 13 & 54 \\ 1 & 0 & -8 & -33 \\ 0 & 0 & -1 & -4 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|c} 0 & 1 & 13 & 54 \\ 1 & 0 & -8 & -33 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Continued: $$\left[\begin{array}{ccc|c} 0 & 1 & 13 & 54 \\ 1 & 0 & -8 & -33 \\ 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{R_2 + 8R_3} \left[\begin{array}{ccc|c} 0 & 1 & 13 & 54 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 - 13R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$\therefore x = 2, y = -1, z = 4$

Since D has a solution, $D \neq \emptyset$. Since these are the same solutions as those of A, and $A \neq \emptyset$, $A = D$, $A \subseteq D$ and $D \subseteq A$. ∎

(d) At this point, you should have proven that the three elementary operations did not change the set of solutions of the linear system $\mathcal{A}$. The key insights of your proofs didn't rely on $\mathcal{A}$'s having exactly three equations and three variables and having coefficients 3, 2, 11 and so forth. We asked you to think about a simple, concrete example so that you could focus on the elementary operations and not be distracted by notation. Now we are asking you to face the notation and prove something about a general linear system.

Consider a system of m linear equations with n variables

$$\mathcal{A}: \quad \left\{ \sum_{j=1}^{n} a_{ij}x_j = b_i \qquad 1 \leq i \leq m \right.$$

where the coefficients a_{ij} and constants b_i are all real numbers. Let $\mathcal{B}$ be the linear system where the i_0th equation of $\mathcal{A}$ has been multiplied by $\alpha \neq 0$.

Assume that both $\mathcal{A}$ and $\mathcal{B}$ have solutions. Prove that $\mathcal{A}$ and $\mathcal{B}$ have the same set of solutions.

This part is worth zero points. Your proof will not be graded. If you got stuck, or wrote a proof you aren't quite sure of, please post to piazza or come to office hours.

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Since $A \neq \emptyset$ and $B \neq \emptyset$, the solutions of $\mathcal{A}$ are contained in $\mathcal{B}$ where the i_0th equation of $\mathcal{A}$ has been multiplied by $\alpha \neq 0$. This means that $A \subseteq B$ since the equation was only scaled by the scalar α and thus the solution is the same. But if you multiply the equation by $\frac{1}{\alpha}$, you will get the same equation, therefore all solutions of $\mathcal{A}$ are in $\mathcal{B}$ and vice versa. Thus, $A = B$ $B \subseteq A$, $A \subseteq B$. ∎

(e) Consider writing down the augmented matrix of a linear system $\mathcal{A}$ and applying elementary row operations to the matrix until you have a matrix in reduced row echelon form (RREF). Let $\mathcal{R}$ be a linear system represented by that RREF matrix. Explain why $\mathcal{A}$ and $\mathcal{R}$ have the same set of solutions. *There's a lot of white space here but this isn't to indicate that you need to write at length to answer this question. You should be able to write your explanation w/ 100 or fewer words (in the space above the horizontal line).* Note that R is the solution set to $\mathcal{R}$, A is the solution set to $\mathcal{A}$.

Since $\mathcal{R}$ is simply the result of $\mathcal{A}$ after undergoing scalar addition, scalar subtraction, and scalar multiplication of equations contained in $\mathcal{A}$, and since these operations are applied to both sides of the equations in $\mathcal{A}$. Thus, the resulting $\mathcal{R}$ would also be contained in $\mathcal{A}$.

Because $\mathcal{R}$ is contained in $\mathcal{A}$ anything that solves $\mathcal{R}$ will also solve $\mathcal{A}$, meaning both $A \neq \emptyset$ and $R \neq \emptyset$, thus $R \subseteq A$.

However, since the RREF of $\mathcal{A}$ is just a reduced form of $\mathcal{A}$ but made easier to solve using elementary row operations that leave the RREF of $\mathcal{A}$ in the same vector space as $\mathcal{A}$ – the RREF essentially contains the same equations as $\mathcal{A}$, but made easier to solve. Thus, R will contain all the solutions of A, $R \subseteq A$ and vice versa.

Proof that the RREF of $\mathcal{A}$ (or $\mathcal{R}$), and $\mathcal{A}$ have the same solution set:

Let N be the inversible matrix $B = B_1 \ldots B_n$ where $B_1 \ldots B_n$ are the elementary row operations done to $\mathcal{A}$ such that $B\mathcal{A} = \mathcal{R}$ where $\mathcal{R}$ is the RREF of $\mathcal{A}$.
Let $X \in \text{null}(A)$. Therefore $\mathcal{A}X = 0$, and thus, $B\mathcal{A} = 0$, meaning $X \in \text{null}(B\mathcal{A})$.
Therefore $B\mathcal{A}X = 0$. By multiplying both sides by B^{-1}, we get $BX = 0$ meaning $X \in \text{null}(A)$ and thus, $\mathcal{R}$ and $\mathcal{A}$ have the same solution set. ∎

Question 2:

One of the goals of this question is to get you into good shape for True/False questions on exams.

For questions (b), (c), and (d) below, please assume the "natural" vector addition and scalar multiplication. For example, for functions assume the addition and multiplication given in section 4.2 of Medici. The ones given in the "An Unusual Vector Space" example are completely valid but are considered "unnatural" for this problem.

(a) Prove the following lemma by showing that all the vector space axioms hold. It's very useful and once you've proven it you can use it whenever you want.

Lemma: Let $\mathbf{V}$ be a vector space over the real numbers. If $\mathbf{V}_0 \subseteq \mathbf{V}$ contains $\mathbf{0}$ and, with the same vector addition and scalar multiplication of $\mathbf{V}$, is closed under vector addition and scalar multiplication then $\mathbf{V}_0$ is a vector space over the real numbers.

I will assume the lemma is true. This means that V_0 is assumed to be a subset of V, which is given as a vector space. Therefore, to prove V_0 is a vector space, I will prove that it is a subspace of V, following subspace test version 1, assuming that the other vector space axioms would already be satisfied by V being a vector space. The subspace test proceeds as follows:

"<u>AI</u>: Closure under vector addition- for $\vec{x}, \vec{y} \in V_0$, $\vec{x}+\vec{y} \in V_0$."

↳ By definition, V_0 is closed under vector addition. Therefore, AI holds for V_0.

"<u>AII</u>: for $\vec{x}, \vec{y}, \vec{z} \in V_0$, $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$".

↳ By definition, because V_0 is a subset of V, with the same vector addition and scalar multiplication, this property stays true for $\vec{x}, \vec{y}, \vec{z} \in V_0$, where $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$. Therefore, AII holds for V_0.

"<u>AIII</u>: there exists a $\vec{0} \in V_0$, such that $\vec{x}+\vec{0}=\vec{x}$ for all $\vec{x} \in V_0$."

↳ By the lemma, there exists a $\vec{0} \in V_0$. following normal vector addition, by definition, for any $\vec{x} \in V_0$, $\vec{x}+\vec{0}=\vec{x}$. Therefore, AIII holds for V_0 //

"<u>AIV</u>: For each $\vec{x} \in V$, there exists a $-\vec{x} \in V$ such that $\vec{x}+(-\vec{x})=\vec{0}$."

↳ By definition, V_0 follows normal scalar multiplication as in V. As proven in class, $(-1)\vec{x}$ for $\vec{x} \in V_0$ is equal to $-\vec{x}$, the additive inverse of $\vec{x} \in V_0$. Therefore, for the scalar -1, $\vec{x}+(-1)\vec{x} = \vec{x}+(-\vec{x})=\vec{0}$. Therefore AII holds for V_0. //

"<u>MI</u>: for $\vec{x}, \vec{y} \in V_0$, $\vec{x}+\vec{y} \in V_0$; closure under vector addition.":

↳ By the lemma, V_0 is closed under vector addition, therefore, MI holds in V_0.

"<u>MII</u>: for all $\vec{x} \in V_0$ and $\alpha, \beta \in \mathbb{R}$, $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$."

↳ By definition, V_0 follows the normal scalar multiplication as does in V, which <u>is</u> a vector space. Therefore, for any $\alpha, \beta \in \mathbb{R}$, $\vec{x} \in V_0$, $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$ in V_0. Therefore, MII holds in V_0.

"<u>MIII</u>: for all $\vec{x}, \vec{y}, \vec{z} \in V_0$, and $\alpha, \beta \in \mathbb{R}$, $(\alpha+\beta)\vec{x}=\alpha\vec{x}+\beta\vec{x}$ and $\alpha(\vec{x}+\vec{y}) = \alpha\vec{x}+\alpha\vec{y}$.

↳ By definition, V_0 follows the normal scalar multiplication as does in V which <u>is</u> a vector space. Therefore, for any $\alpha, \beta \in \mathbb{R}$, $\vec{x}, \vec{y} \in V_0$, $\alpha(\vec{x}+\vec{y}) = \alpha\vec{x}+\alpha\vec{y}$ and $\vec{x}(\alpha+\beta) = \alpha\vec{x}+\beta\vec{x}$. Therefore, MIII holds for V_0.

"<u>MIV</u>: for all $\vec{x} \in V_0$, $\alpha \in \mathbb{R}$, $\alpha\vec{x} \in V_0$; closure under scalar multiplication."

↳ By the lemma, V_0 is closed under scalar multiplication, Therefore, MIV holds for V_0.

Therefore, all 8 axioms are satisfied by $V_0 \subseteq V$, given the lemma is true. Thus, V_0 is a vector space over $\mathbb{R}$. //

∴ QED

True or False: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

Indicate your final answers by **filling in exactly one circle** for each part below (unfilled ○ filled ●).

Hint: if you think any of the following are vector spaces, ask yourself whether there's a way of proving that this is true without having to start at the definition and proving that all eight axioms hold.

(b) The set **V** of all nonpositive real-valued functions on $[2, 3]$ with the usual addition and scalar multiplication is a vector space over $\mathbb{R}$. *Note: a real-valued function on* $[2, 3]$ *is a function* $f : [2, 3] \to \mathbb{R}$. *That is, its domain is* $[2, 3]$ *and its range is a subset of* $\mathbb{R}$. *Its range could be all of* $\mathbb{R}$, *of course. A function is nonpositive if* $f(x) \leq 0$ *for all* x *in its domain.*

○ True

● False

The set V fails by axiom MI: there is <u>no</u> closure under scalar multiplication.

" <u>MI</u>: for all $\alpha \in \mathbb{R}$, $\vec{v} \in V$, $\alpha\vec{v} \in V$; closure under scalar multiplication."

↳ By definition, V is defined for non-positive real-valued functions, meaning their range is always less than or equal to zero. However, if these functions were multiplied by $\alpha \in \mathbb{R}$, where $\alpha < 0$, then for $f(x) \in V$, $\alpha f(x) \geq 0$ because both $\alpha < 0$ and $f(x) \leq 0$. This would make $\alpha f(x)$ false under the terms of V, for $f(x) \neq 0$, $f(x) < 0$, and therefore, there is no closure under scalar multiplication because $f(x)$ would no longer be a non-positive function.

For example, say $\alpha = -1$ and $f(x) = -x^2$: $\alpha f(x) = (-1)(-x^2) = x^2$, which is ≥ 0 for all $x \in \mathbb{R}$, and <u>not</u> in V.

Because MI fails, then V fails to follow <u>ALL</u> 8 axioms, and therefore, is <u>NOT</u> a vector space. // $\therefore$ Q.E.D.

(c) The set **V** of all real polynomials of degree exactly n with the usual addition and scalar multiplication is a vector space. *Note: A "real polynomial" is a polynomial whose coefficients are all real numbers.*

○ True

● False

The set V of all polynomials of order exactly n, is not a vector space because it fails under AIII, because $\vec{0} \notin V$.

- "<u>AIII</u>: there exists a $\vec{0} \in V$ such that $\vec{x} + \vec{0} = \vec{x}$ for $\vec{x} \in V$."

↳ This is false for polynomials of degree 1 or more. If a polynomial has degree <u>exactly</u> n, then it cannot have polynomials within its set of less than degree n, which includes the zero polynomial.
For example, if V_2 is the set of all polynomials with degree exactly 2, then the polynomial $p(x) = 0$, the $\vec{0}$ for polynomials, would not exist because it has degree 0.

Therefore, because AIII fails for V, V is <u>not</u> a vector space over $\mathbb{R}$ because it fails to meet all the axioms. //

$\therefore$ $\not{Q}$.E.D.

True or False: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

Indicate your final answers by **filling in exactly one circle** for each part below (unfilled ○ filled ●).

Hint: if you think the following is a vector space, ask yourself whether there's a way of proving that this is true without having to start at the definition and proving that all eight axioms hold.

(d) The set of 2×2 upper triangular real matrices with the usual entry-wise addition and scalar multiplication is a vector space over the real numbers.

Notes: An $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is upper triangular if $a_{ij} = 0$ for all $1 \leq j < i \leq n$. A matrix is real if all of its entries are real numbers. All integer matrices and all rational matrices are also real matrices. After doing this problem, ask yourself how your answer would apply to $n \times n$ upper triangular matrices.

● True

○ False

NOTE: For a 2x2 upper triangular matrix, the first row's second column's entry is zero.

Let ${}^2\mathbb{R}^2$ be the vector space comprised of all 2x2 matrices with scalar entries $\in \mathbb{R}$. Let the set described in d) be denoted as V. V is a subset of ${}^2\mathbb{R}^2$, considering it is comprised of upper triangular matrices with dimensions 2x2.
V follows the same vector addition and scalar multiplication as in ${}^2\mathbb{R}^2$, and $V \subseteq {}^2\mathbb{R}^2$.
V to be a vector space over $\mathbb{R}$, it must also satisfy the following conditions:

1: "there exists a zero vector $\vec{0} \in V$."

↳ In V, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ can be classified as an upper triangular matrix. Following A III, any $\vec{x} \in V$, $\vec{x} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$, $a, c, d \in \mathbb{R}$ can be written $\vec{x} + \vec{0} = \vec{x}$.
Using $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+0 & 0+0 \\ c+0 & d+0 \end{bmatrix}$, and by A III, this equals $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} = \vec{x}$. ∴, S I is satisfied. //

2: "closed under vector addition; for $\vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} \in V$."

↳ If an upper triangular matrix, $\vec{x} \in V$, $\vec{x} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$, $a, c, d \in \mathbb{R}$ was added to $\vec{y} \in V$, $\vec{y} = \begin{bmatrix} e & 0 \\ f & g \end{bmatrix}$, $e, f, g \in \mathbb{R}$, then $\vec{x} + \vec{y} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \begin{bmatrix} e & 0 \\ f & g \end{bmatrix}$ $= \begin{bmatrix} a+e & 0+0 \\ c+f & d+g \end{bmatrix} = \begin{bmatrix} h & 0 \\ i & j \end{bmatrix}$, for some $h, i, j \in \mathbb{R}$, where $h = a+e$, $i = c+f$, $j = d+g$. By definition, this would satisfy the condition of V, considering how the first row's second column's entry is zero, and the result is still an upper triangular matrix, meaning $\vec{x} + \vec{y} \in V$. //

3: "closed under scalar multiplication; for $\alpha \in \mathbb{R}$, $\vec{x} \in V$, $\alpha \vec{x} \in V$".

↳ Given a scalar $\alpha \in \mathbb{R}$, $\vec{x} \in V$, $x = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$, $a, c, d \in \mathbb{R}$, $\alpha \vec{x} = \alpha \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & 0 \\ \alpha c & \alpha d \end{bmatrix}$. Because both $\alpha \in \mathbb{R}$ and $a, c, d \in \mathbb{R}$, $\alpha a, \alpha c, \alpha \in \mathbb{R}$ as well.
This satisfies the conditions for V, as the first row's second column has entry zero, and the result is an upper triangular matrix.
This means $\alpha \vec{x} \in V$. //

Therefore the conditions of the lemma in part a) have been satisfied, for $V \subseteq {}^2\mathbb{R}^2$, where V is composed of the same vector addition and scalar multiplication as in vector space ${}^2\mathbb{R}^2$: closed under vector addition and scalar multiplication, and the existence of a $\vec{0}$ vector $\in V$.
By this lemma, V, being the set of 2x2 upper triangular matrices, a subset of vector space ${}^2\mathbb{R}^2$, is a vector space over $\mathbb{R}$. //

∴ QED