MAT185 – Linear Algebra
Assignment 3

Instructions:

Please read the **MAT185 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

Full Name: Sara Parvaresh Rizi

Student number: 1010913451

Full Name: Jessica (Xuezheng) Fu

Student number: 1011048607

I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I have not used generative AI in writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By submitting this assignment to Gradescope, I agree that the statements above are true.

Question 1:

Consider the real vector space $V = \text{span}\{x^2e^x, xe^x, e^x\}$. *No justification needed for parts (a)–(d); just fill in the boxes. :)*

(a) Let $T : V \to V$ be the linear transformation corresponding to differentiation: $T(v) = v'$. Using the basis $\alpha = \{x^2e^x, xe^x, e^x\}$ for the domain and the codomain, what matrix $[T]_{\alpha\alpha}$ represents this linear transformation?

$$[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{2} & \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{1} & \boxed{1} \end{bmatrix}$$

(b) Demonstrate that your matrix is correct by computing $[T]_{\alpha\alpha}[v]_\alpha$ for $v = 3x^2e^x - 2xe^x + 6e^x$ to find $[T(v)]_\alpha = [3x^2e^x + 4xe^x + 4e^x]_\alpha$.

$$[T]_{\alpha\alpha}[v]_\alpha = \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{2} & \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{1} & \boxed{1} \end{bmatrix} \begin{bmatrix} \boxed{3} \\ \boxed{-2} \\ \boxed{6} \end{bmatrix} = \begin{bmatrix} \boxed{3} \\ \boxed{4} \\ \boxed{4} \end{bmatrix} = [T(v)]_\alpha$$

(c) What is $[T]_{\alpha\alpha}^{-1}$ for your matrix?

$$[T]_{\alpha\alpha}^{-1} = \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{-2} & \boxed{1} & \boxed{0} \\ \boxed{2} & \boxed{-1} & \boxed{1} \end{bmatrix}$$

(d) Using your $[T]_{\alpha\alpha}^{-1}$, find $v \in V$ such that $v'(x) = x^2e^x$.

$$[T]_{\alpha\alpha}^{-1}[v]_\alpha = \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{0} \\ \boxed{-2} & \boxed{1} & \boxed{0} \\ \boxed{2} & \boxed{-1} & \boxed{1} \end{bmatrix} \begin{bmatrix} \boxed{1} \\ \boxed{0} \\ \boxed{0} \end{bmatrix} = \begin{bmatrix} \boxed{1} \\ \boxed{-2} \\ \boxed{2} \end{bmatrix} = [T^{-1}(v)]_\alpha$$

$v(x)=$ $\boxed{x^2e^x - 2xe^x + 2e^x}$

(e) You've learnt that a function has infinitely many antiderivatives. Why is it that this method has found only one of them?

The coordinate matrix $P_{\alpha\alpha}$ and $P_{\alpha\alpha}^{-1}$ are invertible, meaning they have a reduced row echelon form of I, because they are the change of basis matrices of the common basis α.
This means they can <u>only</u> provide a unique solution for x, when in the form $Ax=b$, and cannot have infinite solutions because their null space is just $\{0\}$. Because of this fundamental property of invertible matrices, it can only present the antiderivative with $+0$. //

Question 2:

(a) What is a parametrized curve that represents the line segment that connects (x_0, y_0) to (x_1, y_1)? That is, what is an $\ell(t)$ such that $\ell(0) = (x_0, y_0)$, $\ell(1) = (x_1, y_1)$, and $\{\ell(t) \mid 0 \le t \le 1\}$ is a line segment?

The parametrized line is $\ell(t) = (1-t)(x_0, y_0) + t(x_1, y_1)$, for $0 \le t \le 1$

(b) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T((x, y)) = (2x + 3y, -6x + y)$. What is the image of the above line segment $\{\ell(t) \mid 0 \le t \le 1\}$ under this linear transformation?

To find the image, I'll find the kernel and dimension of the kernel first, to determine the dimension of the image:

$\ker T = \{T(v) = 0 \mid v \in \mathbb{R}^2\} \to \begin{matrix} 2x+3y=0 \\ -6x+y=0 \end{matrix} \to \left[\begin{array}{cc|c} 2 & 3 & 0 \\ -6 & 1 & 0 \end{array}\right] \xrightarrow{R_2+3R_1} \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 10 & 0 \end{array}\right] \xrightarrow{R_2/10} \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1/2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \to$ both $\begin{matrix} x=0 \\ y=0 \end{matrix} \to \ker T = \{0\}$.

Therefore, $\ker T = \{0\}$ and $\dim \ker T = 0$. Using the dimension formula, and the fact that $\mathbb{R}^2$ is a finite dimensional vector space,

$\dim \operatorname{im} T + \dim \ker T = \dim \text{domain} \to \dim \operatorname{im} T = \dim(\text{line in } \mathbb{R}^2) = 1$.

Considering how $\operatorname{im} T \subseteq \text{codomain} = \mathbb{R}^2$ and if $\dim \operatorname{im} T = 1$, $\operatorname{im} T$ must be a line in $\mathbb{R}^2$.

Therefore $\operatorname{im} T$ is a line segment in $\mathbb{R}^2$, for $0 \le t \le 1$, where $T(\ell(t)) = (2(x_0 + t(x_1 - x_0)) + 3(y_0 + t(y_1 - y_0)), -6(x_0 + t(x_1 - x_0)) + (y_0 + t(y_1 - y_0)))$. //

(c) Consider a general linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$; that is, $T((x, y)) = (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$ where $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$. Let $\ell \subseteq \mathbb{R}^2$ be a line in the plane. In ten words or less, what is the image of ℓ? That is, what is $T(\ell)$?

The image of ℓ ($T(\ell)$) is a line, or a point.

(d) *No justification is needed. You can select more than one answer.* Let $P \subseteq \mathbb{R}^2$ be a polygon[1] and consider a general linear transformation as in part (c). The image of P under T could be

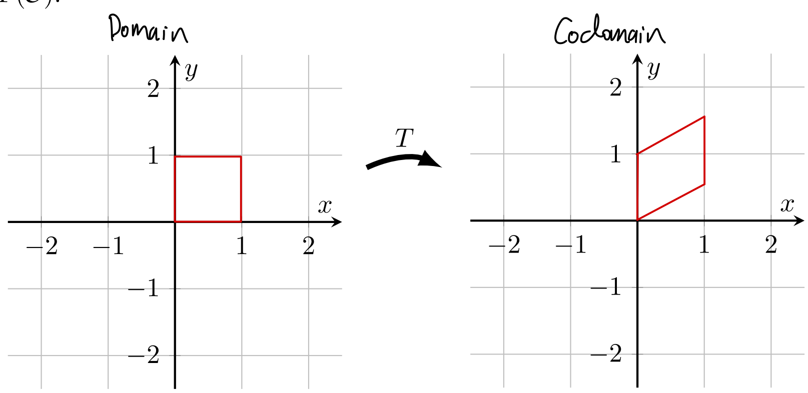
- ○ a circle → that would require $x^2+y^2=r$, I think
- ● a point
- ● a line segment
- ● a polygon
- ○ something else

[1]Here are the types of polygons we mean here. Think about both regular and irregular ones, please.

Question 3:

In the following, you will consider a sequence of linear transformations. For each linear transformation you will asked to draw the image of a square under the linear transformation. *No justification needed for parts (a)–(f); just fill in the boxes. :)*

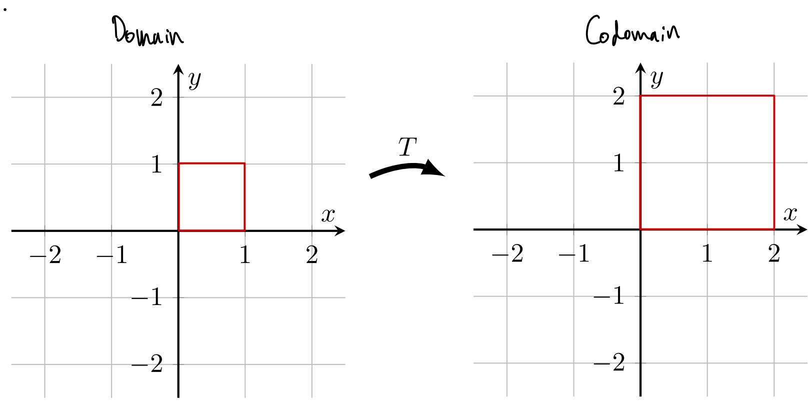
(a) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (x, x/2 + y)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{1}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{1} & \boxed{0} \\ \boxed{1/2} & \boxed{1} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{1}$

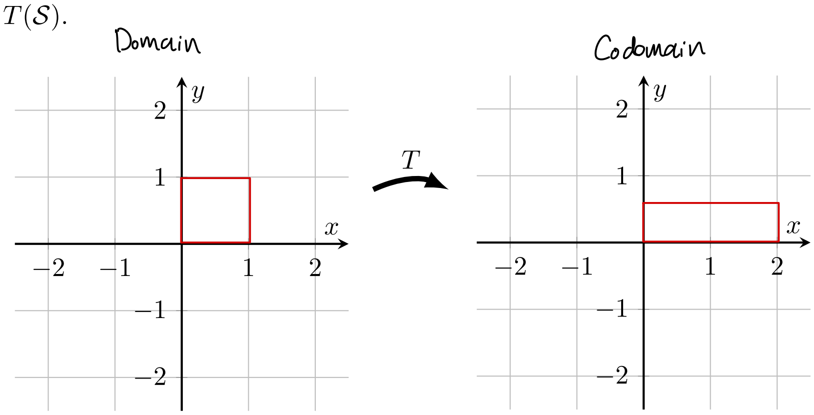
(b) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (2x, 2y)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{4}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{2} & \boxed{0} \\ \boxed{0} & \boxed{2} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{4}$

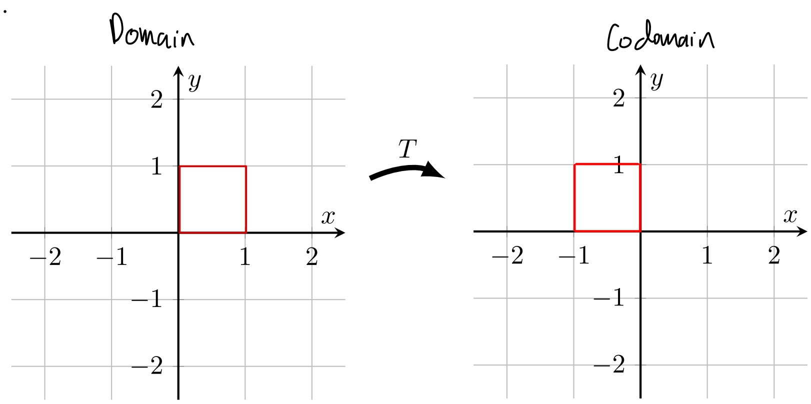
(c) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (2x, y/2)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{1}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{2} & \boxed{0} \\ \boxed{0} & \boxed{1/2} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{1}$

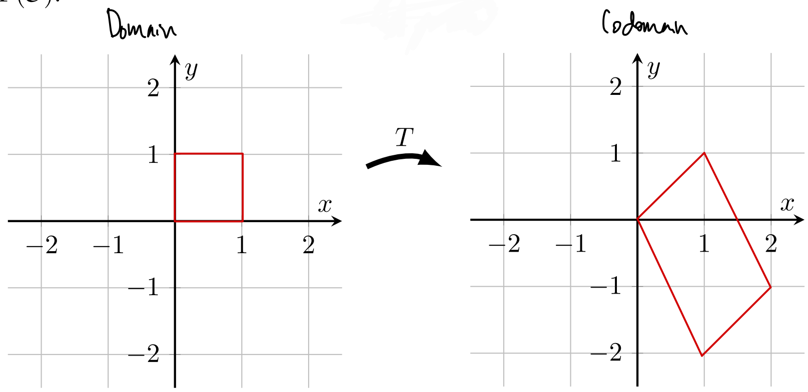
(d) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (-y, x)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{1}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{0} & \boxed{-1} \\ \boxed{1} & \boxed{0} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{1}$

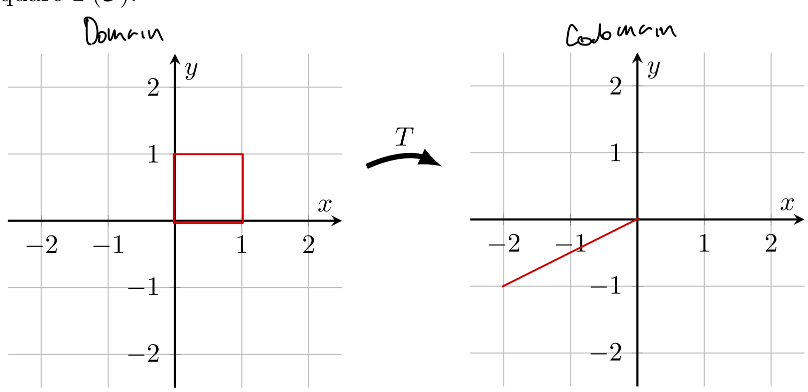
(e) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (x + y, x - 2y)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{3}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{-2} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{-3}$

(f) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T((x, y)) = (-x - y, -x/2 - y/2)$. In the domain, draw the unit square $\mathcal{S} = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. In the codomain draw the image of the unit square $T(\mathcal{S})$.



What is the area of $\mathcal{S}$? What is the area of its image $T(\mathcal{S})$? If α is the standard basis for $\mathbb{R}^2$, what is the matrix $[T]_{\alpha\alpha}$ that represents the linear transformation T? What is $\det[T]_{\alpha\alpha}$?

Area of $\mathcal{S} = \boxed{1}$ Area of $T(\mathcal{S}) = \boxed{0}$ $[T]_{\alpha\alpha} = \begin{bmatrix} \boxed{-1} & \boxed{-1} \\ \boxed{-1/2} & \boxed{-1/2} \end{bmatrix}$ $\det[T]_{\alpha\alpha} = \boxed{0}$

(g) Continuing with the unit square $\mathcal{S}$, as shown in parts (a)–(f), the area of $T(\mathcal{S})$ and $\det[T]_{\alpha\alpha}$ are related. What is the relationship you observed? Prove this relationship holds for a general linear transformation T. *See question 2(c) for what we mean by a general linear transformation. Your answer should not need to refer to the entries of T; there should be no a_{12} or the like in your answer.*

The relationship I observed was that $\|\det([T]_{\alpha\alpha})\| * (\text{Area of } S) = \text{Area of } T(S)$, or rather, given that Area of $S = 1$, Area of $T(S) = \|\det([T]_{\alpha\alpha})\|$.

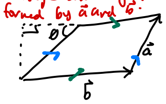
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation from $\mathbb{R}^2$, containing a unit square, to $\mathbb{R}^2$ with the transformed unit square, where the transformation is represented by $[T]_{\alpha\alpha}$, with respect to the standard basis α for both the domain and codomain. Let $[T]_{\alpha\alpha}$ be represented by $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

If we consider the vertices of a unit square, which are $(0,0), (1,0), (0,1), (1,1)$, and we apply T to these vertices, we achieve:

1. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
2. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$
3. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$
4. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}+a_{12} \\ a_{21}+a_{22} \end{bmatrix}$

This result is a parallelogram, represented by vectors $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$. This is because, the vector from $(0,0)$ to (a_{12}, a_{22}) is $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$, from $(0,0)$ to (a_{11}, a_{12}) is (a_{11}, a_{12}), and from (a_{12}, a_{22}) and (a_{11}, a_{21}) to $(a_{11}+a_{12}, a_{21}+a_{22})$ is $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$ respectively, meaning 2 pairs of sides are represented by $\begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$ or $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$, confirming that the result is a parallelogram. (NOTE: here, a parallelogram may have height or width 0, so a line or point could also be produced, depending on $a_{11}, a_{12}, a_{21}, a_{22}$).

image showing parallelogram formed by $\vec{a}$ and $\vec{b}$.



The area of a parallelogram can be obtained by the cross product of its vectors, i.e., if it is made from $\vec{a}$ and $\vec{b}$, $\vec{a} \times \vec{b} = \|\vec{a}\|\|\vec{b}\|\sin\theta$ gives the area of the parallelogram (see image to the right), and because area is positive, we take $\|\vec{a} \times \vec{b}\|$. However, there is no "true" cross product in $\mathbb{R}^2$, considering how only 2 vectors can be orthogonal to each other in $\mathbb{R}^2$, and cannot be to a third. Thus, consider the cross product of two $\mathbb{R}^3$ vector with zero in their z-coordinates: $(a_{11}, a_{21}, 0)$ and $(a_{12}, a_{22}, 0)$, the vectors making up the parallelogram in question. $(a_{11}, a_{21}, 0) \times (a_{12}, a_{22}, 0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \end{vmatrix}$. We observe that for $\vec{a} \times \vec{b}$ in $\mathbb{R}^3$, if the z-coordinates were zero, then the non-zero "remainder" of the cross-product would be the cofactor expansion of $\hat{k}$, where $\vec{a} \times \vec{b} = \hat{k} \cdot \det(A)$, where A is the matrix with the x and y coordinates only, meaning $A = [T]_{\alpha\alpha}$. This is because the expansions with $\hat{i}$ and $\hat{j}$ would have columns of zero, from the z-coordinates, resulting in values of zero. This yields an area equal to $a_{11}a_{22} - a_{21}a_{12}$, which is also what is achieved through $\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}$.

Considering how $\|\hat{k}\| = 1$, as it is a unit vector, and the area of a parallelogram is positive, we can conclude that the Area of $T(S) = \|\det([T]_{\alpha\alpha})\|$. //

QED.

(h) If you started with a square of area 4, $\mathcal{S}_1 = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le 2\}$ how would you use the determinant to compute the area of $T(\mathcal{S}_1)$ for a general linear transformation?

I would construct $[T]_{\alpha\alpha}$ for $T: \mathbb{R}^2 \to \mathbb{R}^2$, apply it to S_1, and calculate $\|\det([T]_{\alpha\alpha})\|$. Then, I would multiply the value of $\|\det([T]_{\alpha\alpha})\|$ by <u>4</u> to get the area of $T(S_1)$. The reasoning behind this is that because the initial shape is a unit square of area 4 instead of 1 like in the previous examples, we'd have to scale the resulting parallelogram area by the initial value as well, because that is its "starting point". As well, this is like multiplying each column of $[T]_{\alpha\alpha}$ by 2, and if $A = \|\det([T]_{\alpha\alpha})\|$, then factoring out 2 from the 2x2 matrix gives $\det(cA) = c^n \det A$, where $n=2$, which gives $2^2 \det A$, as c in this case is 2, which gives $4\|\det([T]_{\alpha\alpha})\|$. //

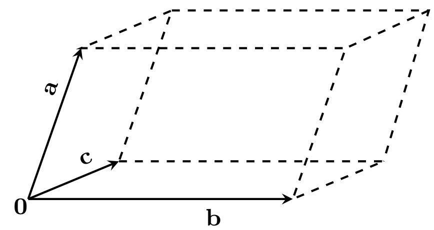
The following question 4 is worth zero points. Your work will not be graded. The question generalizes the previous question and is within your reach if you are comfortable with dot products and cross products. It is material that you may be assumed to already know when you're in AER210. We encourage you to think about and work on this problem and we are happy to talk about it with you, but it is provided "for the curious" in that you will not be tested on it.

YOU MUST UPLOAD THIS PAGE EVEN IF YOU WROTE NOTHING ON IT.

Question 4:

What about volumes of parallelepipeds?

Each face of a parallellepiped is parallel to the opposite face; the parallelepiped is determined by the three edges coming out from any one corner. In the image, we have a corner at $(0,0,0)$ and the three edges are represented using the vectors in $\mathbf{a}, \mathbf{b}, \mathbf{c} \in {}^3\mathbb{R}$, for example the vectors based at the origin $(0,0,0)$.



To find the volume of a parallelepiped[2], you need to find the area of one face and the distance of the opposite face to that one face. For example, if you found the area of the bottom face (the one determined by **b** and **c**) you would need to find the height of the parallelepiped. You would then multiply the two areas.

(a) Using the above figure, explain why the volume of the parallelipiped is $|\ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\ |$. Explain how this is related to the determinant of the matrix whose rows are given by **a**, **b**, and **c**:

$$\det \begin{bmatrix} \mathbf{a}^{\mathbf{T}} \\ \mathbf{b}^{\mathbf{T}} \\ \mathbf{c}^{\mathbf{T}} \end{bmatrix}.$$

a) Let the vectors $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, and $c = (c_1, c_2, c_3)$ be the vectors a, b, c (as shown in diagram). Let there be a matrix V such that $V = \begin{bmatrix} a^T \\ b^T \\ c^T \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$.

The base of the parallelepiped is the parallelogram spanned by the vectors b and c. The area can be given by $A_{base} = |b \times c|$, the magnitude of the cross product of the two vectors.

The height of the parallelepiped is the component of a along the normal to the base $(b \times c)$, which can be found using the dot product $h = \frac{|a \cdot (b \times c)|}{|b \times c|}$ since its the projection of the vector $\vec{a}$ onto the normal vector to the base, which is $b \times c$.

To find volume, you multiply the base by the height: $V = (A_{base})(h) = (|b \times c|)\left(\frac{|a \cdot (b \times c)|}{|b \times c|}\right) = |a \cdot (b \times c)|$

$$= a \cdot \left(\det\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - \det\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + \det\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \right)$$

$$= a_1 \cdot \det\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \cdot \det\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \cdot \det\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \quad ①$$

Therefore, $|a \cdot (b \times c)| = a_1 \cdot \det\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \cdot \det\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \cdot \det\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$ ①

At the same time, the determinate of matrix V: $\det(V) = \det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \cdot \det\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \cdot \det\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \cdot \det\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$ ②

It can be seen that ① = ②, and so $|a \cdot (b \times c)| = \det(V) = \det\left(\begin{bmatrix} a^T \\ b^T \\ c^T \end{bmatrix}\right)$

Thus, the relationship between the volume of the parallelipiped $|a \cdot (b \times c)|$ and $\det\left(\begin{bmatrix} a^T \\ b^T \\ c^T \end{bmatrix}\right)$ is that they are equal, or $|a \cdot (b \times c)| = \det\left(\begin{bmatrix} a^T \\ b^T \\ c^T \end{bmatrix}\right)$.

[2]A parallelepiped is a solid object with corners and edges and an interior. We can understand it using Cartesian coordinates the moment we say where the origin is and where the x, y, and z axes are. In the picture we're implicitly taking the view that the positive x axis is in the same direction as **b**, the positive y axis is in the same direction as **c**, and the positive z axis is in the same direction as **a**. But we need to be careful to remember that edges are collections of points like $(0,0,0)$ and $(1,0,0)$ and so forth while the vectors **a**, **b**, and **c** are in ${}^3\mathbb{R}$.

YOU MUST UPLOAD THIS PAGE EVEN IF YOU WROTE NOTHING ON IT.

(b) Explain why you would have gotten the same volume if you had based your calculation on the front of the parallelepiped (determined by **a** and **b**) or if you had based your calculation on the side of the parallelepiped (determined by **a** and **c**).

$$\text{volume of parallelepiped} = \det \begin{bmatrix} \mathbf{c}^{\mathbf{T}} \\ \mathbf{a}^{\mathbf{T}} \\ \mathbf{b}^{\mathbf{T}} \end{bmatrix} = \det \begin{bmatrix} \mathbf{b}^{\mathbf{T}} \\ \mathbf{c}^{\mathbf{T}} \\ \mathbf{a}^{\mathbf{T}} \end{bmatrix}.$$

b) You would have the same volume if you had based your calculation on the front of the parallelepiped or on the side because you would also be doing the same calculations by finding the area through a cross-product, and then the height by the projection of the remaining vector onto that cross-product.

YOU MUST UPLOAD THIS PAGE EVEN IF YOU WROTE NOTHING ON IT.

(c) Consider the unit cube $\mathcal{C} = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. What is the image of $\mathcal{C}$? (What is $T(\mathcal{C})$ geometrically?) If α is the standard basis for $\mathbb{R}^3$ and $[T]_{\alpha\alpha}$ is the matrix that represents the linear transformation, what is det $[T]_{\alpha\alpha}$ and how does it relate to the volume of $T(\mathcal{C})$?

c) The cube C has to be spanned by the standard basis vectors of $\mathbb{R}^3$, namely, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Let these vectors be denoted as $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Thus, any point in the cube C can be expressed as: $P = xe_1 + ye_2 + ze_3$.

Applying the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ on point P, $T(P) = xT(e_1) + yT(e_2) + zT(e_3)$, we can thus notice how the image of $T(C)$ is a parallelepiped since $T(C) = \{ xT(e_1) + yT(e_2) + zT(e_3) \mid 0 \le x, y, z \le 1\}$. – $T(C)$ is geometrically a parallelepiped.

If $[T]_{\alpha\alpha}$ is the matrix that represents the linear transformation, then, as the vectors are vectors along the sides of the parallelepiped, then, as proven in part a, $\det[T]_{\alpha\alpha}$ will give the volume of the parallelepiped. The magnitude of $\det[T]_{\alpha\alpha}$ or $|\det[T]_{\alpha\alpha}|$ is equal to the volume of $T(C)$ as proven in part a.

(d) If you started with a cube of volume 8, $\mathcal{C}_1 = \{(x, y, z) \mid 0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2\}$, how would you use the determinant to compute the volume of $T(\mathcal{C}_1)$?

Since, as in part c, Volume of $T(C_1) = V_0 |\det[T]_{\alpha\alpha}|$, where V_0 is the original volume of the cube, in this case, $V_0 = 8$. Thus, $T(C_1) = 8 \cdot |\det[T]_{\alpha\alpha}|$ Thus, multiplying the magnitude of the determinant by 8 will give you the volume of $T(C_1)$.