MAT185 – Linear Algebra
Assignment 4

Instructions:

Please read the **MAT185 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

Full Name: Sara Parvaresh Rizi

Student number: 1010913451

Full Name: Xuezheng (Jessica) Fu

Student number: 1011048607

I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I have not used generative AI in writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By submitting this assignment to Gradescope, I agree that the statements above are true.

Preamble

This assignment explores two applications for eigenvectors and eigenvalues: (i) the PageRank algorithm and (ii) solving linear constant-coefficient second-order ordinary differential equations.

In Questions 1 and 2, you will explore using eigenvectors to calculate the ranking of nodes in a directed graph, which is useful for ranking websites by how important they are, e.g., to show a sorted list of websites based on a search criterion. The main problem with ranking websites is that not every website is connected to every other website.[1] We are introducing a technique which is the basis for the PageRank-algorithm, which is the main algorithm used by Google to sort its search results in the 90s and is still in use today. Bryan and Leise [1] describe this algorithm in more detail and it is a great insight if you are interested in more technical details. Reading this publication is not necessary to solve the problems in this assignment. However, we have uploaded the publication to Quercus for the curious.

In Question 3 we are using eigenvalues and eigenvectors to solve an ordinary differential equation (ODE). In that case, we are using a mass-spring system, representing other mechanics ODEs you will solve in the following terms.

The PageRank Algorithm

The PageRank algorithm is an important part of the internal functionality of web search engines. It was established by Google and its founders Sergey Brin and Larry Page in the 1990s. Its main functionality is based on their publication in Reference [2]. We present here a key idea behind the PageRank algorithm, which involves eigenvectors, eigenvalues, and directed graphs. You will work through a small example. In reality, one needs to apply efficient numerical algorithms to estimate the PageRank eigenvector.

As an example, the directed graph in Figure 1 represents a small network of four websites. This graph has 4 *nodes*, each representing a website. If a website links to another website, this is shown by a directed edge. For example, website #1 has two links: one to website #2 and one to website #3. Website #2 also has two links: one to website #3 and one to website #4. We will only consider graphs for which every website links to some other website and no website links to itself. (Although we encourage the curious to explore such situations once they've worked through Question 1 and 2.)

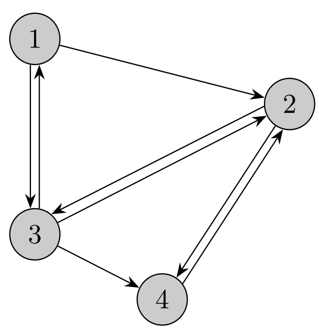


Figure 1: Directed graph, representing a small network of linked websites.

One ranking strategy would be to use the number of incoming hyperlinks. If lots of other websites are linking to one website, surely that one website must be very useful and reliable, right? This strategy is flawed because it can be easily manipulated by someone who creates lots of spurious websites all with hyperlinks pointing to the one website they want to boost up in the ranking. When creating a ranking, not all hyperlinks should carry the same weight!

A hyperlink from a low-ranked website should have less of a contribution to a website's ranking than a hyperlink from a high-ranked website. Separately, there should be some sort of dilution effect: if a website has two hyperlinks pointing to other websites, those two links should count for more (on the receiving ends) than if the website had ten hyperlinks pointing to other websites.

[1] A similar problem would be if you consider a tournament (e.g. a chess tournament), in which not every player is playing against every other player. How can we now calculate a ranking of the best players if we only have insufficient data?

The PageRank approach encapsulates these three ideas:

- The rank of a website is the sum of the weights of all incoming hyperlinks.
- The higher the ranking of a website, the higher the weights are of its outgoing hyperlinks.
- The more outgoing hyperlinks a website has, the lower the weight each of those hyperlinks has.

With this weighing strategy, we calculate the PageRank for each site as follows: The PageRank p_i of website i is a positive scalar and is the weighted sum of the PageRanks of all the websites that have hyperlinks pointing to it:

$$p_i = \sum_{j \to i} \frac{p_j}{N_j} \tag{1}$$

where $j \to i$ denotes there is a hyperlink from site j to site i, p_j is the PageRank of website j, and N_j is the total number of outgoing hyperlinks from site j. For example, to calculate the PageRank of website 2 in Figure 1, there is one incoming hyperlink from site 1 (which has 2 outgoing links), one from site 3 (which has 3 outgoing links), and one from site 4 (which has only one outgoing link). And so, the PageRank of p_2 is defined as

$$p_2 = \frac{1}{2}p_1 + \frac{1}{3}p_3 + p_4. \tag{2}$$

In Question 1, you will find the PageRank of every website shown in Figure 1. In Question 2, you will continue studying the PageRank model for this network and develop tools that will work for much, much larger networks.

References

[1] K. Bryan and T. Leise, *The $25,000,000,000 Eigenvector: The Linear Algebra behind Google*, SIAM Review, 48:3, 569-581, 2006, [online].

[2] S. Brin and L. Page, *The anatomy of a large-scale hypertextual Web search engine*, Computer Networks and ISDN Systems, 30:1-7, 107-117, 1998.

Question 1:

(a) Equation (2) determines the PageRank of node 2. Give the system of PageRank equations for the four nodes in Figure 1, as defined by Equation (1). *No justification is required.*

The system of equations is:
$$\begin{cases} p_1 = \frac{1}{3}p_3 \\ p_2 = \frac{1}{2}p_1 + \frac{1}{3}p_3 + p_4 \\ p_3 = \frac{1}{2}p_1 + \frac{1}{2}p_2 \\ p_4 = \frac{1}{2}p_2 + \frac{1}{3}p_3 \end{cases}$$

(b) Formulate the linear system you found in part (a) as an eigenproblem. Identify the matrix $\mathbf{A}$, the eigenvalue and the eigenvector. *Note: you'll solve for the eigenvectors in the next part of this question.*

The linear system is:
$$\begin{bmatrix} 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/3 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

The matrix A is $\begin{bmatrix} 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/3 & 0 \end{bmatrix}$ in the eigenproblem $Av = \lambda v$.

The eigenvector is $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$, with eigenvalue $\lambda = 1$.

(c) Solve the eigenproblem you formulated in part (b). Calculate the PageRank vector $\mathbf{p}$ for the graph in Figure 1. You will need to use that the components of a PageRank vector must add up to 1.

To simplify calculations I'll bring $[A|\vec{p}]$ to reduced row eschelon form. to make it easier to solve.

$$\left[\begin{array}{cccc|c} 0 & 0 & 1/3 & 0 & p_1 \\ 1/2 & 0 & 1/3 & 1 & p_2 \\ 1/2 & 1/2 & 0 & 0 & p_3 \\ 0 & 1/2 & 1/3 & 0 & p_4 \end{array}\right] \xrightarrow{\substack{3\times R_1,\ 2\times R_3 \\ 6\times R_2,\ 6R_4}} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3p_1 \\ 3 & 0 & 2 & 6 & 6p_2 \\ 1 & 1 & 0 & 0 & 2p_3 \\ 0 & 3 & 2 & 0 & 6p_4 \end{array}\right] \xrightarrow[R_2-2R_1]{R_4-2R_1} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3p_1 \\ 3 & 0 & 0 & 6 & 6p_2-6p_1 \\ 1 & 1 & 0 & 0 & 2p_3 \\ 0 & 3 & 0 & 0 & 6p_4-6p_1 \end{array}\right] \xrightarrow[R_2/3]{R_4/3} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3p_1 \\ 1 & 0 & 0 & 2 & 2p_2-2p_1 \\ 1 & 1 & 0 & 0 & 2p_3 \\ 0 & 1 & 0 & 0 & 2p_4-2p_1 \end{array}\right]$$

$$\xrightarrow{R_3-R_4} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3p_1 \\ 1 & 0 & 0 & 2 & 2p_2-2p_1 \\ 1 & 0 & 0 & 0 & 2p_3-2p_4+2p_1 \\ 0 & 1 & 0 & 0 & 2p_4-2p_1 \end{array}\right] \xrightarrow{R_2-R_3} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3p_1 \\ 0 & 0 & 0 & 2 & 2p_2-4p_1-2p_3+2p_4 \\ 1 & 0 & 0 & 0 & 2p_3-2p_4+2p_1 \\ 0 & 1 & 0 & 0 & 2p_4-2p_1 \end{array}\right] \xrightarrow{R_2/2} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3p_1 \\ 0 & 0 & 0 & 1 & p_2-2p_1-p_3+p_4 \\ 1 & 0 & 0 & 0 & 2p_3-2p_4+2p_1 \\ 0 & 1 & 0 & 0 & 2p_4-2p_1 \end{array}\right] \xrightarrow{\text{swap rows to get } I} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2p_3-2p_4+2p_1 \\ 0 & 1 & 0 & 0 & 2p_4-2p_1 \\ 0 & 0 & 1 & 0 & 3p_1 \\ 0 & 0 & 0 & 1 & p_2-2p_1-p_3+p_4 \end{array}\right]$$

Now we have equations $\begin{cases} \boxed{1}\ p_1 = 2p_3-2p_4+2p_1 \\ \boxed{2}\ p_2 = 2p_4-2p_1 \\ \boxed{3}\ p_3 = 3p_1 \\ \boxed{4}\ p_4 = p_2-2p_1-p_3+p_4 \end{cases}$, and given that $p_1+p_2+p_3+p_4 = 1\ \boxed{5}$, we can rewrite

we can solve for $\vec{p}$, yielding $\vec{p} = \begin{bmatrix} 2/25 \\ 2/5 \\ 6/25 \\ 7/25 \end{bmatrix}$. //

(d) Which website of the graph in Figure 1 has the highest and which has the lowest PageRank?

Website1 has PageRank $\frac{1}{3}P_3 = 2/25$.

Website 2 has PageRank $\frac{1}{2}P_1 + \frac{1}{3}P_3 + P_4 = \frac{1}{25} + \frac{2}{25} + \frac{7}{25} = \frac{10}{25}$

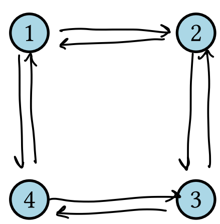
Website 3 has PageRank $\frac{1}{2}P_1 + \frac{1}{2}P_2 = \frac{2}{25} + \frac{5}{25} = \frac{7}{25}$

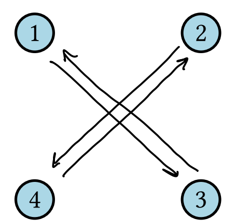
Website 4 has PageRank $\frac{1}{2}P_2 + \frac{1}{3}P_3 = \frac{5}{25} + \frac{2}{25} = \frac{7}{25}$.

Therefore, website 1 has the lowest, and website 2 the highest PageRank. //

(e) Think about other possible networks. What would be a network that would lead to all four webpages having the same PageRank? Add the hyperlinks to the figure below on the left. What would be a *different* network would lead to all four webpages having the same PageRank? Add the hyperlinks to the figure on the right. *No justification is required.*

There are more than two possible answers to this question. Please make sure that your hyperlinks are clearly legible in your scan, including the direction they're pointing.





Question 2:

Now that you are more familiar with the eigenproblem from Question 1, you're going to study the eigenvalues and eigenvectors of the matrix $\mathbf{A}$ from Question 1.

(a) Compute $\mathbf{Ax}$ where $\mathbf{x} \in {}^4\mathbb{R}$. How is $\sum_{i=1}^{4}(\mathbf{Ax})_i$ related to $\sum_{i=1}^{4} x_i$?

The calculation for Ax is:

$$Ax = \begin{bmatrix} 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/3\, x_3 \\ 1/2\, x_1 + 1/3\, x_3 + x_4 \\ 1/2\, x_1 + 1/2\, x_2 \\ 1/2\, x_2 + 1/3\, x_3 \end{bmatrix}$$

The calculation for $\sum_{i=1}^{4}(Ax)_i$ is $\frac{1}{3}x_3 + \frac{1}{2}x_1 + 1/3\,x_3 + x_4 + 1/2\,x_1 + 1/2\,x_2 + 1/2\,x_2 + 1/3\,x_3 = x_3 + x_1 + x_4 + x_2$.

The calculation for $\sum_{i=1}^{4} x_i$ is $x_1 + x_2 + x_3 + x_4$.

The two above calculations are related, as the result of $\sum_{i=1}^{4} x_i$ equals that of $\sum_{i=1}^{4}(Ax)_i$. //

Assume $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ. What can you say about λ and/or $\mathbf{x}$?

If x is an eigenvector of A with eigenvalue λ, $Ax = \lambda x$, by definition. If we sum both sides, $\sum_{i=1}^{4}(Ax)_i = \sum_{i=1}^{4}\lambda x_i = \lambda\sum_{i=1}^{4} x_i$. from above, we've established that $\sum_{i=1}^{4}(Ax)_i = \sum_{i=1}^{4} x_i$, therefore, $\lambda\sum_{i=1}^{4} x_i = \sum_{i=1}^{4} x_i$. This means $(\lambda - 1)\sum_{i=1}^{4} x_i = 0$, so either $\sum_{i=1}^{4} x_i = 0$, in other words, the sum of components of x equal zero, or $\lambda = 1$. //

(b) Using your computation of $\mathbf{Ax}$ where $\mathbf{x} \in {}^4\mathbb{R}$, how is $\sum_{i=1}^{4}|(\mathbf{Ax})_i|$ related to $\sum_{i=1}^{4}|x_i|$? *You will want to use the triangle inequality: if $a, b \in \mathbb{R}$ then $|a+b| \le |a| + |b|$.*

The calculation for $\sum_{i=1}^{4}|(Ax)_i|$ is: $|\frac{1}{3}x_3| + |\frac{1}{2}x_1 + 1/3\,x_3 + x_4| + |1/2\,x_1 + 1/2\,x_2| + |1/2\,x_2 + 1/3\,x_3| \le |\frac{1}{3}x_3| + |\frac{1}{2}x_1| + |\frac{1}{3}x_3| + |x_4| + |\frac{1}{2}x_1| + |\frac{1}{2}x_2| + |\frac{1}{2}x_2| + |\frac{1}{3}x_3|$ which, in turn, equals $|x_1| + |x_2| + |x_3| + |x_4|$

The calculation for $\sum_{i=1}^{4}|x_i|$ is: $|x_1| + |x_2| + |x_3| + |x_4| \ge |x_1 + x_2 + x_3 + x_4|$

The relationship between the two is that $\sum_{i=1}^{4}|(Ax_i)| \le |x_1| + |x_2| + |x_3| + |x_4|$ <u>and</u> $\sum_{i=1}^{4}|(x_i)| = |x_1| + |x_2| + |x_3| + |x_4|$. More specifically, though, $\sum_{i=1}^{4}|(Ax_i)| \le |x_1| + |x_2| + |x_3| + |x_4| = \sum_{i=1}^{4}|(x_i)|$, meaning $\sum_{i=1}^{4}|(Ax)_i|$ is less than or equal to $\sum_{i=1}^{4}|(x_i)|$. //

If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, what can you say about its eigenvalue λ?

If x is an eigenvector of A with eigenvalue λ, by definition, $Ax = \lambda x$.
As well, we know $\sum_{i=1}^{4}|(Ax)_i| \le \sum_{i=1}^{4}|(x_i)|$ from above. This can be rewritten as $\sum_{i=1}^{4}|\lambda x_i| \le \sum_{i=1}^{4}|x_i|$, by definition, and so, $|\lambda|\sum_{i=1}^{4}|x_i| \le \sum_{i=1}^{4}|x_i|$, meaning $|\lambda| \le 1$. //

(c) Combining your answer to (a) and (b) what can you say about $\mathbf{A}$'s eigenvalues and eigenvectors?

A has at least one eigenvalue equal to 1, and all eigenvalues satisfy $|\lambda| \le 1$. Eigenvectors corresponding to $\lambda = 1$ may have their sum equal to zero, but all those corresponding to eigenvalues that aren't 1 must have their sum equal to zero. //

(d) Using matlab or python or wolframalpha or whatever, what are the eigenvalues of $\mathbf{A}$? Please give the eigenvalues in order of decreasing magnitude, largest to smallest. Remember, $|a + ib| = \sqrt{a^2 + b^2}$. Round to two decimal places. *No justification is required.*

$\lambda_1 \approx$ $\boxed{1}$ $\lambda_2 \approx$ $\boxed{-0.601+0.221i}$ $\lambda_3 \approx$ $\boxed{-0.601-0.22i}$ $\lambda_4 \approx$ $\boxed{0.203}$

Is $\mathbf{A} \in {}^4\mathbb{R}^4$ diagonalizable? ○ Yes ● No

Is $\mathbf{A} \in {}^4\mathbb{C}^4$ diagonalizable? ● Yes ○ No

(e) Let $\mathbf{x} \in {}^4\mathbb{C}$ be a linear combination of the eigenvectors of $\mathbf{A}$; that is, $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + c_4\mathbf{x}_4$ where $c_i \in \mathbb{C}$. What is $\mathbf{A}\mathbf{x}$? What is $\mathbf{A}^2\,\mathbf{x}$? What is $\mathbf{A}^{100}\,\mathbf{x}$? *Your answers should include the symbols λ_i when referring to eigenvalues, rather than the approximate values you found for eigenvalues.*

If $x \in {}^4\mathbb{C}$, $x = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$, $Ax = A(c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4) = c_1Ax_1 + c_2Ax_2 + c_3Ax_3 + c_4Ax_4 = c_1\lambda_1x_1 + c_2\lambda_2x_2 + c_3\lambda_3x_3 + c_4\lambda_4x_4$.

Then, $A^2x = A \cdot Ax = A(c_1\lambda_1x_1 + c_2\lambda_2x_2 + c_3\lambda_3x_3 + c_4\lambda_4x_4) = c_1\lambda_1Ax_1 + c_2\lambda_2Ax_2 + c_3\lambda_3Ax_3 + c_4\lambda_4Ax_4 = c_1\lambda_1^2x_1 + c_2\lambda_2^2x_2 + c_3\lambda_3^2x_3 + c_4\lambda_4^2x_4$.

Therefore, if this proceeds even more times until $A^{100}x$, because $c_i\lambda_i$ are just constants and $Ax_i = \lambda x_i$, $A^{100}x = c_1\lambda_1^{100}x_1 + c_2\lambda_2^{100}x_2 + c_3\lambda_3^{100}x_3 + c_4\lambda_4^{100}x_4$. //

As $n \to \infty$, what does $\mathbf{A}^n\,\mathbf{x}$ do, and why?

As $n \to \infty$, $\lim_{n\to\infty} A^nx = \lim_{n\to\infty}(c_1\lambda_1^nx_1 + c_2\lambda_2^nx_2 + c_3\lambda_3^nx_3 + c_4\lambda_4^nx_4)$. Because $\lambda_2, \lambda_3, \lambda_4$ from part d have magnitudes less than 1, they will act like geometric sequences and $c_2\lambda_2^nx_2, c_3\lambda_3^nx_3, c_4\lambda_4^nx_4 \to 0$ as $n \to \infty$. However, $\lambda_1 = 1$, and so, $\lim_{n\to\infty} 1^n = 1$. Therefore, $\lim_{n\to\infty} A^nx = \lim_{n\to\infty}(c_1\lambda_1^nx_1 + c_2\lambda_2^nx_2 + c_3\lambda_3^nx_3 + c_4\lambda_4^nx_4) = c_1\lambda_1^nx_1 = c_1x_1$. //

(f) You have the matrix $\mathbf{A}$ and you would like to find an approximation of an eigenvector for the eigenvalue 1. You have access to matlab and python but not to subroutines that compute eigenvalues and eigenvectors.

Playing around, you discover that if you choose a random vector $\mathbf{x} \in {}^4\mathbb{R}$ and you compute $\mathbf{A}^n\,\mathbf{x}$ for larger and larger values of n then $\mathbf{A}^n\,\mathbf{x}$ seems to stop changing (so far as the computer can see). If you call that computational limit $\mathbf{x}_\infty$, is $\mathbf{x}_\infty$ (approximately) an eigenvector for $\mathbf{A}$ with eigenvalue 1?

● Yes ○ No

If yes, why? If no, why not?

Yes, because this implies that for large n, $A^nx \approx \lambda^nx$, which follows from the calculations did in part e. x would be an approximation for the eigenvector of A, because it associates with the "dominant" eigenvalue of A, being λ, because it is implied that $\lim_{n\to\infty} A^nx = \lim_{n\to\infty}(\lambda^nx + \lambda_0^nx_0 + \ldots) \cong \lambda^nx$.
As well, λ would have to be 1, or else $\lim_{n\to\infty} \lambda^nx$ would either be zero if $|\lambda| < 1$, or would diverge to ∞ if $\lambda < -1, \lambda > 1$.
Therefore, x would be an approximation of the real eigenvector of A, with $\lambda = 1$. //

For the curious: The linear system in question 1(a) is a small linear system: four equations in four unknowns. Your immediate reaction upon seeing the system should have been to set up an augmented matrix, do some elementary row operations, and find all solutions. Finding solutions by viewing it as an eigenproblem seems pretty artificial! And even if you do decide to take the eigenproblem view of things; you can find the null space of $\mathbf{I} - A$ via row reduction. So row reduction must be the right approach, right?

For a internet search engine, how big is the matrix $\mathbf{A}$? Its size depends on the number of considered webpages, which is roughly on the order of tens of billions for a search engine. We do know that the $\mathbf{A}$ is very sparse because, on average, there are fewer than 100 links per webpage pointing to different page. So, $\mathbf{A}$ is a sparse matrix of size 10 billion $\times$ 10 billion. The number of operations needed to transform an $n \times n$ matrix to reduced row echelon form is $\mathcal{O}(n^3)$. But the number of operations needed to compute $\mathbf{Ax}$ once is $\mathcal{O}(n^2)$. When you have $\mathbf{Ax}$, computing $\mathbf{A}^2\mathbf{x}$ is a matter of computing $\mathbf{A}(\mathbf{Ax})$ — this will be another $\mathcal{O}(n^2)$. Computing $\mathbf{A}^k\,\mathbf{x}$ is $\mathcal{O}(kn^2)$ and so, if k isn't large, this will be faster than computing the reduced row echelon form. Using this approach to approximate an eigenvector for the largest eigenvalue is called *power method*. It works well as long as the geometric multiplicity of the largest eigenvalue is 1. The speed with which it converges has to do with how big the gap is between $|\lambda_1|$ and $|\lambda_2|$ where λ_2 is an eigenvalue with "next largest" size. The bigger $|\lambda_1| - |\lambda_2|$ is, the faster the power method converges. You can find Python implementations of the power method to calculate the dominant eigenvalue and eigenvector online.

Numerical methods for computing eigenvalues have been an area of active research and continue to be so. The power method is a snowflake on top of an iceberg when it comes to the topic!

A separate question is: what properties of a network ensure that 1 will be an eigenvalue of the matrix $\mathbf{A}$? What properties ensure that its eigenspace will be one dimensional?

Question 3:

The typical ODEs in mechanics are second-order ODEs, because the acceleration is the second derivative of the position in respect to time. If the ODEs are linear, they can be reformulated as two linear first-order ODEs. If the ODEs have constant coefficients and if the forcing is constant, we can solve the system using the methods from this course.

In this problem, you will will explore solving a second-order ODE which describes a mass-spring system using the techniques discussed in this course. *For the curious: the methods used in this problem work equally well if there is damping in the system and these methods apply for other oscillatory systems such as RL circuits and RLC circuits.*

Consider the following mass-spring system with one mass and two springs:

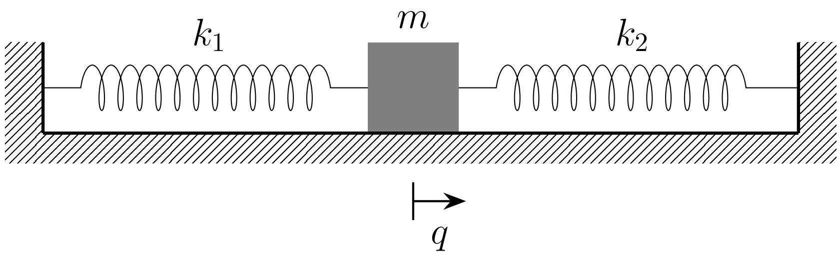


Figure 2: Mass-spring system with the mass m and two springs of stiffness k_1 and k_2.

We will ignore the effect of gravity and of friction and we will assume the displacements are small enough that Hooke's law applies. Let m be the mass, and k_1 and k_2 be the spring constants (i.e. the stiffnesses) of the springs. The equation of motion for the system in Figure 2 is

$$m\ddot{q}(t) = -k_1 q(t) - k_2 q(t) = -(k_1 + k_2)q(t). \tag{3}$$

Here, q is the displacement from rest; if the system is at rest then $q(t) = 0$ for all time t. To understand the force: a displacement of the mass m in positive q direction result in the left spring to lengthen, which leads to pulling in $-q$ direction, and the right spring to compress, which leads to pushing in $-q$ direction.

(a) Using the substitution of $x_1(t) = q(t)$ and $x_2(t) = \dot{q}(t)$, show that Equation (3) can be expressed as the following system of two first-order ODEs:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1+k_2}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{A}\mathbf{x}(t). \tag{4}$$

Let $x_1(t) = q(t)$, and $x_2(t) = \dot{q}(t)$. Thus, $\dot{x}_1(t) = \dot{q}(t) = x_2(t)$, and $\dot{x}_2(t) = \ddot{q}(t)$.

Since $m\ddot{q}(t) = -(k_1+k_2)q(t)$ can be rewritten as $\ddot{q}(t) = \frac{-(k_1+k_2)}{m} q(t)$:

$$\dot{x}_2(t) = \ddot{q}(t) = -\frac{(k_1+k_2)}{m} q(t) = -\frac{(k_1+k_2)}{m} x_1(t).$$

Thus, we have the equations: $\dot{x}_1(t) = x_2(t)$ and $\dot{x}_2(t) = \frac{-(k_1+k_2)}{m} x_1(t)$

which can be rewritten as $\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{(k_1+k_2)}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A x(t)$ where $A = \begin{bmatrix} 0 & 1 \\ -\frac{(k_1+k_2)}{m} & 0 \end{bmatrix}$, as required.

You will now solve Equation (3), using Equation (4) and the methods introduced in this course.

(b) View $\mathbf{A}$ as a complex matrix, rather than a real one, $\mathbf{A} \in {}^2\mathbb{C}^2$, and compute the eigenvalues of $\mathbf{A}$ from Equation (4). To keep your computations simple, replace $k_1 + k_2$ with k (the effective spring constant). Your eigenvalues should be of the form $\pm i\omega$ where ω is a function of k and m. The expression you will find for ω is called the "natural angular frequency" or the "natural frequency". For the rest of Question 3, refer to ω rather than to expressions involving k and m.

Given equation 4: $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ where $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}$

To compute the eigenvalues: $C_A(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 \\ \frac{k}{m} & \lambda \end{bmatrix} = 0$

$$(\lambda)(\lambda) - (-1)\left(-\frac{k}{m}\right) = 0$$

$$\lambda^2 + \frac{k}{m} = 0$$

$$\lambda^2 = -\frac{k}{m} \rightarrow \lambda = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}}$$

Since it is given that the eigenvalues should be of the form $\pm i\omega$, and $\lambda = \pm i\sqrt{\frac{k}{m}}$, it is clear that $\omega = \sqrt{\frac{k}{m}}$.

Thus, the eigenvalues of A are $\pm i\omega$ where $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_1+k_2}{m}}$.

(c) Compute the eigenvectors for the two complex eigenvalues calculated in Question 3 (b). ***Hint:*** The eigenvectors will be complex as well. *You will need to rewrite* $\mathbf{A}$ *so that it no longer involves* k *and* m*, only* ω*.*

From part b, the eigenvalues are $i\omega$ and $-i\omega$, and from part a, $A = \begin{bmatrix} 0 & 1 \\ -\frac{(k_1+k_2)}{m} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$

For $\lambda_1 = i\omega$: $(\lambda_1 I - A)v = 0 \rightarrow \begin{bmatrix} i\omega & -1 \\ \omega^2 & i\omega \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

From the system: $(i\omega)v_1 - v_2 = 0 \rightarrow v_2 = (i\omega)v_1$ ①

$\omega^2 v_1 + (i\omega)v_2 = 0 \xrightarrow{(\omega \neq 0)} \omega v_1 + iv_2 = 0 \rightarrow iv_2 = -\omega v_1 \rightarrow v_2 = i\omega v_1 = (i\omega)v_1$ ②

Noticing that equation ① and ② both simplify to $v_2 = (i\omega)v_1$, thus $v_1 = \begin{bmatrix} 1 \\ i\omega \end{bmatrix}$ by choosing $v_1 = 1$.

For $\lambda_2 = -i\omega$: $(\lambda_2 I - A)v = 0 \rightarrow \begin{bmatrix} -i\omega & -1 \\ \omega^2 & -i\omega \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

From the system: $(-i\omega)v_1 - v_2 = 0 \rightarrow v_2 = (-i\omega)v_1$ ③

$\omega^2 v_1 + (-i\omega)v_2 = 0 \rightarrow \omega v_1 + (-i)v_2 = 0 \rightarrow i\omega v_1 - (-1)v_2 = 0 \rightarrow v_2 = (-i\omega)v_1$ ④

From ③ and ④, it can be seen that $v_2 = (-i\omega)v_1$, and so $v_2 = \begin{bmatrix} 1 \\ -i\omega \end{bmatrix}$ by choosing $v_1 = 1$.

$\therefore$ The eigenvector corresponding to $\lambda_1 = i\omega$ is $\begin{bmatrix} 1 \\ i\omega \end{bmatrix}$, and the eigenvector corresponding to $\lambda_2 = -i\omega$ is $\begin{bmatrix} 1 \\ -i\omega \end{bmatrix}$.

(d) Using the eigenvalues and eigenvectors you found, what is the general solution, $\mathbf{x}(t)$, for the system in Equation (4)? What is each component, $x_1(t)$ and $x_2(t)$? *Please keep working with ω rather than k and m. Also, you want to check your work by checking on a piece of scratch paper that $\dot{x}_1(t) = x_2(t)$. There's no need to present that verification here.*

Using the eigenvalues $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$, and the eigenvectors $V_1 = \begin{bmatrix}1\\ i\omega\end{bmatrix}$, $V_2 = \begin{bmatrix}1\\ -\omega\end{bmatrix}$, the general solution for equation 4 is $x(t) = c_1 e^{i\omega t}\begin{bmatrix}1\\ i\omega\end{bmatrix} + c_2 e^{-i\omega t}\begin{bmatrix}1\\ -i\omega\end{bmatrix}$, where c_1 and c_2 are complex constants. //

The component $x_1(t) = c_1 e^{i\omega t}\cdot 1 + c_2 e^{-i\omega t}\cdot 1 = c_1 e^{i\omega t} + c_2 e^{i\omega t}$.

and the component $x_2(t) = c_1 e^{i\omega t}\cdot i\omega + c_2 e^{-i\omega t}(-i\omega) = i\omega c_1 e^{i\omega t} - i\omega c_2 e^{-i\omega t}$.

Thus, $x_1(t) = c_1 e^{i\omega t} + c_2 e^{i\omega t}$, and $x_2(t) = i\omega c_1 e^{i\omega t} - i\omega c_2 e^{-i\omega t}$. //

(e) Assume that the mass is initially at rest, $q(0) = q_0$ and $\dot{q}(0) = 0$. This determines $\mathbf{x}(0)$. What is $\mathbf{x}(t)$? *At this point, we haven't discussed what it means to have a complex number in the exponent but you do still know that $e^0 = 1$.*

Given that $q(0) = q_0$ and $\dot{q}(0) = 0$: $x_1(0) = q(0) = c_1 e^{i\omega(0)} + c_2 e^{i\omega(0)} = c_1 e^0 + c_2 e^0 = c_1 + c_2$ and

$x_2(0) = \dot{q}(0) = 0 = i\omega c_1 e^{i\omega(0)} - i\omega c_2 e^{i\omega(0)} = i\omega c_1 e^0 - i\omega c_2 e^0 = i\omega c_1 - i\omega c_2$.

Thus, $c_1 + c_2 = q_0$ and $i\omega c_1 - i\omega c_2 = 0$.

Since $i\omega c_1 - i\omega c_2 = 0 \rightarrow i\omega(c_1 - c_2) = 0 \rightarrow c_1 = c_2$, thus $c_1 + c_2 = q_0$ can be rewritten as $2c_1 = q_0 \rightarrow c_1 = \frac{q_0}{2}$ or $2c_2 = q_0 \rightarrow c_2 = \frac{q_0}{2}$.

Thus, $c_1 = c_2 = \frac{q_0}{2}$, and substituting into $x(t) = c_1 e^{i\omega t}\begin{bmatrix}1\\ i\omega\end{bmatrix} + c_2 e^{-i\omega t}\begin{bmatrix}1\\ -i\omega\end{bmatrix}$, you get $x(t) = \frac{q_0}{2} e^{i\omega t}\begin{bmatrix}1\\ i\omega\end{bmatrix} + \frac{q_0}{2} e^{-i\omega t}\begin{bmatrix}1\\ -i\omega\end{bmatrix}$ //

(f) At this point, you have an expression for $q(t)$ that involves i. Use Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, to express $q(t)$ in terms of real quantities only. ***For the curious:*** *you can use these ideas to rewrite the general solution from part (d) as something involving* cos *and* sin *if you assume the initial conditions $q(0) = q_0$ and $\dot{q}(0) = v_0$.*

Using $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $q(t) = x_1(t) = c_1 e^{i\omega t} + c_2 e^{i\omega t} = \frac{q_0}{2} e^{i\omega t} + \frac{q_0}{2} e^{-i\omega t}$ since $c_1 = c_2 = \frac{q_0}{2}$,

$q(t)$ can be rewritten as $q(t) = \frac{q_0}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) = \frac{q_0}{2}\left(\cos(\omega t) + i\sin(\omega t) + \cos(\omega t) - i\sin(\omega t)\right) = \frac{q_0}{2}\left(2\cos(\omega t)\right) = q_0\cos(\omega t)$.

Thus, $q(t) = q_0\cos(\omega t)$. //

(g) Check if your found solution for $q(t)$ satisfies the ODE in Equation (3).

Using $q(t) = q_0\cos(\omega t)$, $\dot{q}(t) = -\omega q_0\sin(\omega t)$ and $\ddot{q}(t) = -\omega^2 q_0\cos(\omega t)$ through derivation.

Rewriting equation 3: $m\ddot{q}(t) = -(k_1 + k_2)q(t) \rightarrow \ddot{q}(t) = -\frac{k_1+k_2}{m}q(t) \rightarrow \ddot{q}(t) = -\omega^2 q(t)$, since $\frac{k_1+k_2}{m} = \frac{k}{m} = \left(\sqrt{\frac{k}{m}}\right)^2 = \omega^2$.

To check, substitute $q(t) = q_0\cos(\omega t)$ into the rearranged form of equation 3:

$$\ddot{q}(t) = -\omega^2 q(t) = -\omega^2(q_0\cos(\omega t)) = -\omega^2 q_0\cos(\omega t)$$

Since this matches the derived form of $\ddot{q}(t) = -\omega^2 q_0\cos(\omega t)$, the solution satisfies the ODE in equation 3. //